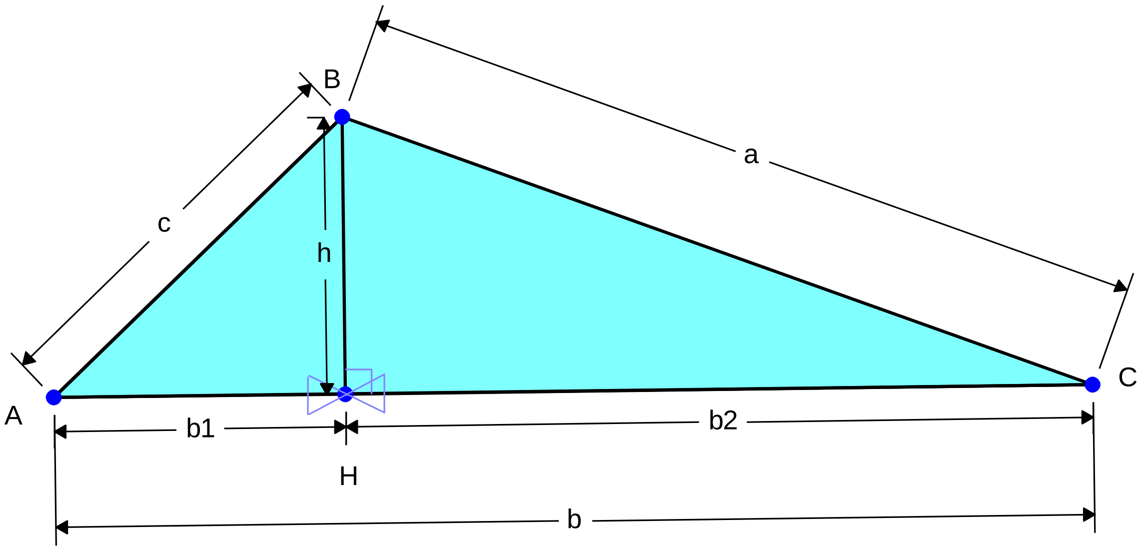
Proving Heron formula

Given triangle ABC,

Its area is $A = \sqrt{S(S-a)(S-b)(S-c)}$

where S is half of perimeter $S = \frac{a+b+c}{2}$



Suppose, we construct the height BH as show, we have

$b_1^2 = c^2 - h^2$ (*)

$b_2^2 = a^2 - h^2$

$b_1^2 - b_2^2 = c^2 - a^2$

$(b_1 - b_2)(b_1 + b_2) = c^2 - a^2$

$(b_1 - b_2)(b) = c^2 - a^2$

Or $b_1 - b_2 = \frac{c^2 - a^2}{b}$

Replace $b_2 = b - b_1$

$$2b_1 = b + \frac{c^2 - a^2}{b} = \frac{b^2 + c^2 - a^2}{b}$$

Or $b_1 = \frac{1}{2}\ \frac{b^2+c^2-a^2}{b}$

From $b_1^2 = c^2 - h^2$ (*) above we get

$$h^2 = c^2 - b_1^2 = (c - b_1)(c + b_1)$$

$$h^2 = \left(c - \frac{1}{2}\ \frac{b^2 + c^2 - a^2}{b}\right)\left(c + \frac{1}{2}\ \frac{b^2 + c^2 - a^2}{b}\right)$$

$$= \frac{(2bc - b^2 - c^2 + a^2)(2bc + b^2 + c^2 - a^2)}{4b^2}$$

$$= \frac{\left(a^2-(b-c)^2\right)\left((b+c)^2-a^2\right)}{4b^2}$$

$$= \frac{(a - b + c)(a + b - c)(b + c - a)(b + c + a)}{4b^2}$$

Rearrange

$$h^2 = \frac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{4b^2}$$

Replace $a + b + c = 2s$

$$h^2 = \frac{(2s)\ (2s - 2a)\ (2s - 2b)\ (2s - 2c)}{4b^2}$$

$$h^2 = \frac{16(s)(s - a)(s - b)(s - c)}{4b^2}$$

$$h = \sqrt{\frac{16(s)(s - a)(s - b)(s - c)}{4b^2}}$$

$$h = 2b\sqrt{s(s - a)(s - b)(s - c)}$$

The area of triangle ABC with the height h is

A = ½ bh = $\frac{1}{2} b * 2b \sqrt{s(s-a)(s-b)(s-c)}$

A = $\sqrt{s(s-a)(s-b)(s-c)}$